APPROXIMATION AND SCHOENFLIES EXTENSION OF C^m-DIFFEOMORPHISMS ($m \geqq 0$)

BY

JEROME L. PAUL[1]

PART I. PRELIMINARIES

1. **Introduction.** The Schoenflies extension theorem (cf. [1], [2], [3]) asserts that a homeomorphism $h: S \to E$ of an $(n-1)$-sphere S in euclidean n-space E is extendable to a homeomorphism $H: \bar{B} \to E$ of the closed n-cell $\bar{B}$ in E bounded by S if, and only if, h is extendable to a homeomorphism h^e of a neighborhood V of S in E. In this paper, we investigate the existence of such extensions H satisfying certain side conditions. We show, for example, (Theorem 1.1 of Part III) that if h^e is a good approximation on V to a given homeomorphism G defined on a neighborhood N of $\bar{B}$, then there exist extensions H of h which are good approximations to G. Furthermore (Theorem 9.1 of Part III), a certain class of homeomorphisms h^e of V into E which are good approximations to G merely on a neighborhood of some *point* of S, admit extensions H which spread this area of approximation to a large portion of the interior of S. In order to study this latter problem, we introduce in Part II a distance $\delta(M, M')$ between topological $(n-1)$-spheres in E, and investigate the relationship between this distance and other well known distances.

2. **Notation and definitions.** We now clarify some basic notation and definitions. Let $E = E^n$, $(n \geqq 2)$, be a euclidean n-space of points $x = (x^2, \ldots, x^n)$, provided with the usual euclidean norm and metric:

$$\|x\| = \left[\sum_{i=1}^{n} (x^i)^2\right]^{1/2}, \qquad d(x, y) = \|x-y\|.$$

The distance between *subsets* A, B of E will have its usual meaning. For c a fixed point of E, and $r > 0$ a constant, we denote the $(n-1)$-sphere and open n-cell about c, of radius r, by

$$S(c, r) = \{x \in E \mid d(x, c) = r\}, \quad B(c, r) = \{x \mid d(x, c) < r\},$$

setting $S = S(0, 1)$, and $B = B(0, 1)$. By a topological $(n-1)$-sphere $\mathscr{M}$ in E, we mean the image in E of S under some homeomorphism h. We say that h *defines* $\mathscr{M}$.

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A *topological* (open) n-cell in E is defined similarly. Corresponding to any constant a, $0 < a < r$, we introduce the n-shell

$$\sigma_a(S(c, r)) = \{x \in E \mid r-a < \|x-c\| < r+a\}.$$

If $\mathscr{M}$ is a topological $(n-1)$-sphere in E, then we denote the bounded component of $E-\mathscr{M}$ by $\dot{J}\mathscr{M}$, and the closure of $\dot{J}\mathscr{M}$ in E by $J\mathscr{M}$. $\mathscr{M}$ will be called elementary if some (hence *every*, as is readily proved) homeomorphism h defining $\mathscr{M}$ is extendable as a homeomorphism into E over an open neighborhood N of S relative to E. This is equivalent (cf. [4]) to requiring that $\mathscr{M}$ is locally flat. A homeomorphism h of S into E which is so extendable will itself be termed elementary. A homeomorphism g into E of a neighborhood N of S will be termed *S-admissible* if g carries points of $JS \cap N$ into points of $Jg(S)$.

If $\mathscr{M}$ is a topological $(n-1)$-sphere in E, and $f: J\mathscr{M} \to E$ is a homeomorphism of $J\mathscr{M}$ into E, then

$$f(\dot{J}\mathscr{M}) = \dot{J}f(\mathscr{M}). \tag{2.1}$$

A proof of (2.1) can be found, for example, in [5].

A C^m-diffeomorphism, $m \geqq 0$, will have its usual meaning. The notion of a "C^m_z-diffeomorphism" will also be useful. Let z be an arbitrary point of E, and let $g: X \to E$ be a homeomorphism of an open neighborhood X of z in E. If $g \mid (X-z)$ is a C^m-diffeomorphism, then g will be called a C^m_z-diffeomorphism.

The identity mapping will be denoted by I, without regard to domain. If A is a subset of E, we will denote the complement $E-A$ of A by CA, the interior of A by $\dot{A}$, the closure of A by $\bar{A}$, and the topological boundary of A by βA.

3. **Some approximation theorems.**

Definition 3.1. Let X be a subset of E^n. A real-valued positive continuous function η, defined on X, will be called a *modulus* on X.

Definition 3.2. Let $H: X \to E^n$ be a continuous mapping, where X is a subset of E^m, and let η be a modulus on X. Then a continuous mapping $F: X \to E^n$ will be called a C^0 *η-approximation to H on X* if

$$d(H(x), F(x)) < \eta(x) \qquad (x \in X). \tag{3.1}$$

Suppose $H: X \to E^n$ is a mapping of class C^1, where X is an open subset of E^m. We introduce the gradients

$$\operatorname{grad} H^i = (\partial H^i/\partial x^1, \ldots, \partial H^i/\partial x^m)$$

of the components H^i, $i = 1, \ldots, n$, of H.

Definition 2.3. Let $H: X \to E^n$ be a mapping of class C^1, and let η be a modulus on X. A mapping $F: X \to E^n$, of class C^1, will be called a C^1 *η-approximation to H on X* if

$$d(H(x), F(x)) + \sum_{i=1}^{n} \|(\operatorname{grad}(H-F)^i)(x)\| < \eta(x) \qquad (x \in X). \tag{3.2}$$

THEOREM 3.1. *Corresponding to a diffeomorphism $H: X \to E^n$ of an open subset X of E^n, there exists a modulus η on X, such that any C^1 η-approximation, F to H on X, is a diffeomorphism of X onto $H(X)$.*

Proof. Cf. proof of Theorem 1.2 in [6].

We now state a "Composition Theorem". This theorem and its proof appears in [6] (cf. Theorem 1.1 and Corollary 3.2 of [6]).

THEOREM 3.2. *Let $F: X \to E^m$ be a diffeomorphism of an open subset X of E^m onto an open set Y of E^m, and let $G: Y \to E^n$ be a C^1-mapping into E^n. Corresponding to an arbitrary modulus ζ on X, there exist moduli τ on X and σ on Y with the following properties.*

If $f: X \to E^m$ is a C^1-mapping of X into E^m which is a C^1 τ-approximation to F, and if $g: Y \to E^n$ is a C^1-mapping of Y into E^n which is a C^1 σ-approximation to G, then the composite function gf maps X into E^n and is a C^1 ζ-approximation to GF on X.

The following trivial proposition will be useful.

PROPOSITION 3.1. *Let X be a subset of E^n, and let $f_1: X \to E^n$ be a homeomorphism such that $f_1^{-1}: f_1(X) \to E^n$ is uniformly continuous. Hence corresponding to an arbitrary positive number ε, there exists a positive number δ such that if $y_1, y_2 \in f_1(X)$ and $d(y_1, y_2) < \delta$, then $d(f_1^{-1}(y_1), f_1^{-1}(y_2)) < \varepsilon$. For such a choice of δ, if $f_2: X \to E^n$ is a homeomorphism which is a C^0 δ-approximation to f_1 on X, and if $A = f_1(X) \cap f_2(X) \neq \varnothing$, then $f_2^{-1}|A$ is a C^0 ε-approximation to $f_1^{-1}|A$.*

Proof. Let $y \in A$. Then $d(y, f_1 f_2^{-1}(y)) = d(f_2 f_2^{-1}(y), f_1 f_2^{-1}(y)) < \delta$. Hence $d(f_1^{-1}(y), f_2^{-1}(y)) = d(f_1^{-1}(f_2 f_2^{-1}(y)), f_1^{-1}(f_1 f_2^{-1}(y))) < \varepsilon$.

THEOREM 3.3. *Let $\mathscr{M}$ be a topological $(n-1)$-sphere in E^n, and let $f: \mathscr{M} \to E^n$ be a homeomorphism. Suppose C is a compact subset of $\dot{J}f(\mathscr{M})$. Setting $\delta = d(f(\mathscr{M}), C)$, if $g: \mathscr{M} \to E^n$ is any homeomorphism which is a C^0 δ-approximation to f on $\mathscr{M}$, then $\dot{J}g(\mathscr{M}) \supset C$. Similarly, if A is any subset of E^n such that $\dot{A} \supset Jf(\mathscr{M})$, and $\eta = d(\beta A, Jf(\mathscr{M}))$, then if $g: M \to E^n$ is any homeomorphism which is a C^0 η-approximation to f on $\mathscr{M}$, then $\dot{A} \supset Jg(\mathscr{M})$.*

The following theorem will be useful in proving Theorem 3.3.

THEOREM 3.4 (BORSUK SEPARATION CRITERION). *Let D be a compact subset of E^n, let x_0 be a point in $E^n - D$, and let $p: (E^n - x_0) \to S^{n-1}$ be defined by*

$$p(x) = (x - x_0)/\|x - x_0\| \qquad (x \in E^n - x_0). \tag{3.3}$$

In order that x_0 lie in the unbounded component of $E^n - D$, it is necessary and sufficient that $p|D$ be inessential (i.e., homotopic to a constant mapping).

A proof of Theorem 3.4 may be found in [7, p. 275].

Proof of Theorem 3.3. To prove the first part of Theorem 3.3, suppose to the contrary that there exists a point $x_0 \in C$ such that $x_0 \notin \dot{J}g(\mathscr{M})$. By our choice of δ, we see that $x_0 \notin g(\mathcal{M})$, and hence x_0 lies in the unbounded component of $E^n - g(\mathcal{M})$. Then setting $D = g(\mathcal{M})$, Theorem 3.4 implies that the mapping p in (3.3) is such that $p|D$ is inessential. Hence there exists a homotopy $K: D \times [0, 1] \to S$ such that

$$K(u, 0) = p(u) \qquad (u \in D), \tag{3.4}$$

and

$$K(u, 1) = c \in S \qquad (u \in D). \tag{3.5}$$

Define a mapping $H: f(\mathcal{M}) \times [0, 1] \to S$ by

$$H(x, t) = p((1-2t)x + 2t \cdot gf^{-1}(x)) \qquad (0 \leqq t \leqq \tfrac{1}{2}), \tag{3.6}$$

$$H(x, t) = K(gf^{-1}(x), 2t-1) \qquad (\tfrac{1}{2} \leqq t \leqq 1). \tag{3.7}$$

Since $d(x, gf^{-1}(x)) < \delta$ for all $x \in f(\mathcal{M})$, we see that the right member of (3.6) is defined. Using (3.4) we see that H is consistently defined. Hence H is a homotopy connecting $p|f(\mathcal{M})$ and a constant mapping, and, since $x_0 \in Jf(\mathcal{M})$, this contradicts Theorem 3.4. This establishes the first part of Theorem 3.3. The second part is proved similarly.

Part II. Various distances between topological $(n-1)$-spheres in E^n

Let M', M'' be two topological $(n-1)$-spheres in E^n. We shall investigate some basic properties relating to three distances, denoted by $\Delta(M', M'')$, $\rho(M', M'')$, and $\delta(M', M'')$, between M' and M''. The first two distances are well known, and have been used, for example, in the study of certain open cell problems (cf. [8]). We shall introduce a third distance $\delta(M', M'')$, which will be used in Part III to study various approximation problems related to the Schoenflies extension theorem. It turns out that all three distances define (nonequivalent) metrics on the set of topological $(n-1)$-spheres in E^n.

First we recall the distance $\Delta(M', M'')$, defined as

$$\Delta(M', M'') = \max \left(\max_{x \in M'} d(x, M''), \max_{x \in M''} d(x, M') \right). \tag{1}$$

One sees that

$$\Delta(M', M'') = \Delta(M'', M'), \tag{2}$$

and

$$\Delta(M', M'') = 0 \Leftrightarrow M' = M''. \tag{3}$$

Moreover, if M, M', M'' are topological $(n-1)$-spheres in E, then

$$\Delta(M, M'') \leqq \Delta(M, M') + \Delta(M', M''). \tag{4}$$

We now recall the Fréchet distance $\rho(M', M'')$. We call a positive number ε *F-admissible* relative to the pair (M', M'') if there exist homeomorphisms $f', f'': S \to E$ such that $f'(S) = M'$, $f''(S) = M''$, and $d(f'(x), f''(x)) < \varepsilon$ for all

$x \in S$. Equivalently, ε is F-admissible relative to (M', M'') if there exists a homeomorphism h' of M' onto M'' such that $d(x, h'(x)) < \varepsilon$ for all $x \in M'$. Set

$$T(M', M'') = \{\varepsilon > 0 \mid \varepsilon \text{ is } F\text{-admissible relative to } (M', M'')\}. \tag{5}$$

Note that $T(M', M'') = T(M'', M')$. The Fréchet distance ρ is defined by

$$\rho(M', M'') = \operatorname{glb} T(M', M''). \tag{6}$$

Then

$$\rho(M', M'') = \rho(M'', M'), \tag{7}$$

$$\rho(M', M'') = 0 \Leftrightarrow M' = M'', \tag{8}$$

and

$$\rho(M, M'') \leqq \rho(M, M') + \rho(M', M''). \tag{9}$$

We now define the distance $\delta(M', M'')$. We call a positive number η *J-admissible* relative to the pair (M', M'') if the following is true: given any pair A', B' of subsets of E such that $\dot{A}' \supset JM'$, $\dot{J}M' \supset \bar{B}'$, and, $\min (d(M', \beta A'), d(M', B')) \geqq \eta$, then $\dot{A}' \supset JM''$ and $\dot{J}M'' \supset \bar{B}'$. Note that any $\eta \geqq \operatorname{diam} (JM' \cup JM'')$ is necessarily J-admissible relative to (M', M''). Set

$$S(M', M'') = \{\eta > 0 \mid \eta \text{ is } J\text{-admissible relative to } (M', M'')\}. \tag{10}$$

Then let

$$\delta(M', M'') = \max (\operatorname{glb} S(M', M''), \operatorname{glb} S(M'', M')). \tag{11}$$

Figure 1 illustrates a case in which $S(M', M'') \neq S(M'', M')$.

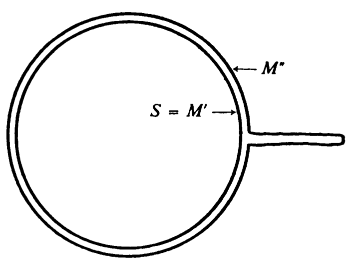

FIGURE 1

REMARK. Clearly, if $\eta \in S(M', M'')$, and $\zeta \geqq \eta$, then $\zeta \in S(M', M'')$. It follows that $S(M', M'')$ either contains, or is contained in, $S(M'', M')$, and

$$\delta(M', M'') = \operatorname{glb} (S(M', M'') \cap S(M'', M')). \tag{12}$$

One sees that

$$\delta(M', M'') = \delta(M'', M'), \tag{13}$$

and

$$\delta(M', M'') = 0 \Leftrightarrow M' = M''. \tag{14}$$

Moreover, if M, M', M'' are topological $(n-1)$-spheres in E, then

$$\delta(M, M'') \leqq \delta(M, M') + \delta(M', M''). \tag{15}$$

To prove (15), we first state the following proposition (α).

(α) *Let M, M' be topological $(n-1)$-spheres in E. Then $\Delta(M, M') < \eta$ for all $\eta \in S(M, M')$.*

A proof of (α) will be given in the verification of (16) which follows.

Proof of (15). It suffices to show that if $\eta \in S(M, M')$, and $\zeta \in S(M', M'')$, then $\eta + \zeta \in S(M, M'')$. Suppose to the contrary that there exists $\eta \in S(M, M')$ and $\zeta \in S(M', M'')$ such that $\eta + \zeta \notin S(M, M'')$. We then have two cases.

Case 1. *There exists a set A such that $\dot{A} \supset JM$, $d(\beta A, M) \geqq \eta + \zeta$, and $\dot{A} \not\supset JM''$.* Suppose $\beta A \cap M'' \neq \varnothing$. Choose a point $x \in \beta A \cap M''$. Since $\zeta \in S(M', M'')$, (α) implies that there exists a point $y \in M'$ such that $d(x, y) < \zeta$. Similarly, since $\eta \in S(M, M')$, there exists a point $z \in M$ such that $d(y, z) < \eta$. Hence $d(x, z) \leqq d(x, y) + d(y, z) < \zeta + \eta$. Therefore, $d(\beta A, M) < \zeta + \eta$, which is a contradiction.

On the other hand, suppose $\beta A \cap M'' = \varnothing$. We then have two possibilities.

(i) $JM'' \cap \beta A = \varnothing$. Then $JM \subset E - \bar{A}$, and $d(M'', M) > d(\beta A, M) \geqq \eta + \zeta$. To see that $d(M'', M) > d(\beta A, M)$, let $y \in M''$. Then the line segment joining y to any point $x \in M$ must meet βA in a point $z \neq y$. Hence $d(y, x) > d(z, x) \geqq \eta + \zeta$, and therefore $d(M'', M) > d(\beta A, M)$. Now by ($\alpha$), we see that $d(M'', M) \leqq d(M'', M') + d(M', M) < \zeta + \eta$, which is a contradiction.

(ii) $\dot{J}M'' \cap \beta A \neq \varnothing$. Then choose a point $x \in \dot{J}M'' \cap \beta A$. Note that $d(\beta A, M') \geqq \zeta$. Now if $x \notin \dot{J}M'$, set $A' = E - x$ and $B' = \varnothing$. Then $\dot{A}' \supset JM'$, $\bar{B}' \subset \dot{J}M'$, and $\min(d(\beta A', M'), d(M', B')) = d(\beta A', M') \geqq \zeta$. Then since $\zeta \in S(M', M'')$, we must have $\dot{A}' \supset JM''$, contradicting our choice of x.

On the other hand, if $x \in \dot{J}M'$, set $A_* = E - x$, and $B_* = \varnothing$. Then $\dot{A}_* \supset JM$, $JM \supset \bar{B}_*$, and $\min(d(\beta A_*, M), d(M, B_*) = d(\beta A_*, M) = d(x, M) \geqq d(\beta A, M) \geqq \eta + \zeta > \eta$. Since $\eta \in S(M, M')$, we must have $\dot{A}_* \supset JM'$, contradicting our choice of x.

Case 2. *There exists a set B such that $\bar{B} \subset \dot{J}M$, $d(M, \bar{B}) \geqq \eta + \zeta$, and $\dot{J}M'' \not\supset \bar{B}$.* We first note that $d(M, \bar{B}) > \zeta$. For if this were not the case, the compactness of $\bar{B}$ and M' would imply the existence of points $b \in \bar{B}$, $y \in M'$ such that $d(y, b) \leqq \zeta$. Then using (α), there exists a point $x \in M$ such that $d(x, y) < \eta$. Hence $d(x, b) \leqq d(x, y) + d(y, b) < \eta + \zeta$, contradicting the fact that $d(M, \bar{B}) \geqq \eta + \zeta$. Note also that since $d(M, \bar{B}) \geqq \eta + \zeta > \eta$, and $\eta \in S(M, M')$, we have $\bar{B} \subset \dot{J}M'$. Setting $A' = E$, $B' = B$, since $\zeta \in S(M', M'')$, we must have $\bar{B}' = \bar{B} \subset \dot{J}M''$, contradicting our assumption. This completes the proof of (15).

We have the following comparison of $\Delta(M', M'')$ and $\delta(M', M'')$:

$$\Delta(M', M'') \leqq \delta(M', M''). \tag{16}$$

To verify (16), suppose $\Delta(M', M'') = \eta > 0$. Then we may assume that there exists a point $x \in M'$ such that $d(x, M'') = \eta$. Suppose first that $x \in \dot{J}M''$. Then setting

$B''=x$, and $A''=E$, we see that $d(M'', B'')=\eta$ and $\dot{J}M''\supset B''$, where as $\dot{J}M'\not\supset B''$. Hence $\eta\notin S(M'', M')$ and so glb $S(M'', M')\geqq\eta$, i.e., $\Delta(M', M'')=\eta\leqq\delta(M', M'')$. Now suppose $x\in CJM''$. Then setting $A''=E-x$, and $B''=\varnothing$, we see that $d(\beta A'', M'')=d(x, M'')=\eta$, and $\mathring{A}''\supset JM''$, whereas $\mathring{A}''\not\supset M'\subset JM'$. Hence $\eta\notin S(M'', M')$ and again $\Delta(M', M'')=\eta\leqq\delta(M', M'')$. This proves (16). Figure 2 illustrates a case where $\Delta(M', M'')<\delta(M', M'')$.

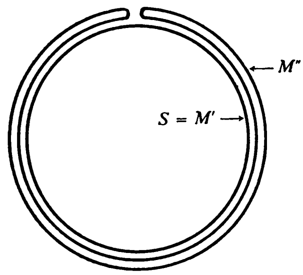

FIGURE 2

We also have the following comparison

(17) $$\delta(M', M'') \leqq \rho(M', M'').$$

To verify (17), note that Theorem 3.3 of Part I implies that

$$T(M', M'') \subset S(M', M'') \cap S(M'', M'),$$

from which (17) follows.

Figure 3 illustrates a case in which $\delta(M', M'')<\rho(M', M'')$.

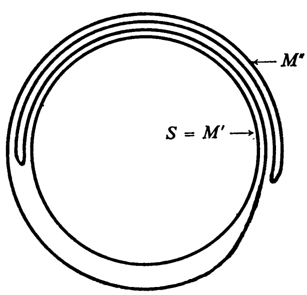

FIGURE 3

REMARKS. We have seen that all three distances Δ, δ, ρ define metrics on the set $\mathscr{S}$ of topological $(n-1)$-spheres in E. If we denote the resulting topological spaces by $\mathscr{S}(\mathscr{T}_\Delta)$, $\mathscr{S}(\mathscr{T}_\delta)$, and $\mathscr{S}(\mathscr{T}_\rho)$, we see from (16) and (17) that

(18) $$\mathscr{T}_\Delta \subseteq \mathscr{T}_\delta \subseteq \mathscr{T}_\rho.$$

Actually, the inclusions in (18) are strict. To see this, we examine Figure 2. It is clear that there exist topological spheres M'' so that $\Delta(S, M'')$ is arbitrarily small, and yet $\delta(S, M'') \geqq \frac{1}{2}$. This shows that $\mathcal{T}_\Delta \subset \mathcal{T}_\delta$. Similarly, an analysis of Figure 3 shows that $\mathcal{T}_\delta \subset \mathcal{T}_\rho$.

Let $E\mathcal{S}$ denote the subset of $\mathcal{S}$ consisting of the *elementary* topological $(n-1)$-spheres in E. The following theorem implies that the C^∞-diffeomorphs of $\mathcal{S}$ form a dense subset of $E\mathcal{S}(\mathcal{T}_\delta)$ (and hence also of $E\mathcal{S}(\mathcal{T}_\Delta)$) when $n \neq 4$.

THEOREM 1. *Let M be an elementary topological $(n-1)$-sphere in E^n $(n \neq 4)$, and let ε be an arbitrary positive number. Then there exists a C^∞-diffeomorph N of S in E such that* $\delta(M, N) < \varepsilon$.

Proof. Since M is elementary, the Schoenflies extension theorem asserts that $\dot{J}M$ is an open topological n-ball. But then $\dot{J}M$ is C^∞-diffeomorphic to $\dot{J}S(0, r)$. This latter statement is a consequence of Theorem 6.3 of [9] for $n=2, 3$, and follows from Theorem 5.1 of [10] for $n \geqq 5$. By taking $r > 1$ and very close to 1, one verifies Theorem 1.

QUESTION 1. Is Theorem 1 true for an *arbitrary* topological $(n-1)$-sphere in E? (i.e., are the elementary topological $(n-1)$-spheres dense in $\mathcal{S}(\mathcal{T}_\delta)$?).

QUESTION 2. Is Theorem 1 true when ρ replaces δ?

REMARK. The answer to Question 2 is affirmative $(n \neq 4)$ if M is defined by a homeomorphism which extends to a *stable* homeomorphism of E (cf. [11] or [12]).

The following theorem can be established using Theorem 1 and an adaptation of the proof of Theorem 9.1 in [8].

THEOREM 2. *Let F be a homeomorphism of E^n onto itself, $n \neq 4$, and let ε be an arbitrary modulus on* $(0, \infty)$. *Then there exists a C^∞-diffeomorphism G of E^n onto itself such that*

$$\delta(F(S(c, t)), G(S(c, t))) < \varepsilon(t) \qquad (t > 0).$$

PART III. APPROXIMATIONS IN THE SCHOENFLIES PROBLEM

1. **Introduction.** Let V be a neighborhood of S in E, and let $f\colon V \to E$ be an S-admissible homeomorphism. In view of the Schoenflies extension theorem, the following question arises: if $g\colon V \to E$ is any homeomorphism which is a good C^0 approximation to f, do $f|S$ and $g|S$ admit extension F and G, respectively, where F and G map $V \cup JS$ into E and are good C^0 approximations to one another? Even more: given the homeomorphism $F\colon V \cup JS \to E$, if $g\colon V \to E$ is any homeomorphism which is a good C^0 approximation to $F|V$, does $g|S$ admit an extension $G\colon V \cup JS \to E$ which is a good C^0 approximation to F? This latter question is answered affirmatively in Theorem 1.1 below. We assume throughout Part III that all homeomorphisms defined in a neighborhood of S in E are S-admissible, and we set $\sigma_a(S) = \sigma_a$.

THEOREM 1.1. *Let $F\colon \sigma_a \cup JS \to E$ be a homeomorphism. Corresponding to any positive number ε, there exists a neighborhood N of S in σ_a, and a positive number δ, with the following properties. If $g\colon \sigma_a \to E$ is any homeomorphism which is a C^0*

δ-approximation to $F|\sigma_a$, then there exists a homeomorphism $G: \sigma_a \cup JS \to E$ which is an extension of $g \mid [N \cup (\sigma_a - JS)]$, and is a C^0 ε-approximation to F on $V \cup JS$.

In §2 below, we shall introduce a special case of Theorem 1.1, namely Theorem 2.1. In §6, we shall derive Theorem 1.1 from Theorem 2.1. The proof of these theorems will involve a detailed analysis of the explicit solution of the Schoenflies extension problem as presented in [3]. Rather than reproduce the proof of this theorem, we refer the reader to [3], and also to the summary given in §§13, 14 of [13]. Our notation throughout will be that of [3], with minor exceptions which will be noted. For example, the n-shell σ_a is written δ_a in [3].

It will be assumed in what follows that $F: \sigma_a \cup JS \to E$ is a given fixed homeomorphism. The proofs of certain lemmas and theorems used to prove Theorem 1.1 will be sketched or omitted entirely. The complete details appear in [14].

2. **A canonical extension of $F \mid [N \cup (\sigma_a - JS)]$.** For convenience of notation, we set $F|\sigma_a = f_1$. Let Q be the "x^n-pole" of S, i.e. Q is the point on S at which x^n assumes its maximum. Let $B(f_1(Q), r)$ be an open n-ball such that $B(f_1(Q), 2r) \subset f_1(\sigma_a)$. Following the techniques of [5], let μ_1 be a C^∞-diffeomorphism of $\bar{B}(f_1(Q), 2r)$ onto E such that

$$\mu_1(y) = y \qquad (y \in B(f_1(Q), r)). \tag{2.1}$$

Set

$$\eta_1(x) = f_1^{-1}\mu_1^{-1}(x) \qquad (x \in E). \tag{2.2}$$

Then define $\phi_1: \sigma_a \to E$ by

$$\phi_1(x) = \eta_1 f_1(x) \qquad (x \in \sigma_a). \tag{2.3}$$

It follows from (2.1) and (2.2) that

$$\phi_1(x) = x \qquad (x \in f_1^{-1}(B(f_1(Q), r))). \tag{2.4}$$

Take an $(n-1)$-sphere S_Q with center at Q and with diameter so small that $JS_Q \subset f_1^{-1}(B(f_1(Q), r))$. Now since ϕ_1, reduces to the identity on a neighborhood of JS_Q, we may proceed as in [3], constructing explicit mappings t, R, T, a and sets K, D, M, H', etc., and, in terms of these objects, an explicit homeomorphism $\Lambda_{\phi_1}: \sigma_a \cup JS \to E$ which is an extension of $\phi \mid [N \cup (\sigma_a - JS)]$, where N is a neighborhood of S in E. Once the mappings t, R, T, a and the sets K, D, M, H', etc., have been chosen, the mapping Λ_{ϕ_1} is completely determined by the procedures in [3]. Note from (2.3) that the homeomorphism $\eta_1^{-1}\Lambda_{\phi_1}: \sigma_a \cup JS \to E$ extends $F \mid [N \cup (\sigma_a - JS)]$. We shall assume in §§2–6 that a fixed choice for the above mappings and sets has been made, and we will denote the resulting (fixed) homeomorphism $\eta_1^{-1}\Lambda_{\phi_1}$ by F^e. We state the following theorem in terms of this fixed mapping F^e.

THEOREM 2.1. *Given the homeomorphism $F: \sigma_a \cup JS \to E$ let F^e be the fixed extension of $F \mid [N \cup (\sigma_a - JS)]$ indicated above. Corresponding to any positive number*

ε, there exists a positive number δ, such that if $g: \sigma_a \to E$ is any homeomorphism which is a C^0 δ-approximation to $F|\sigma_a$, then there exists a homeomorphism $G^e: \sigma_a \cup JS \to E$ which is an extension of $g \mid [N \cup (\sigma_a - JS)]$, and is a C^0 ε-approximation to F^e on $\sigma_a \cup JS$.

The proof of Theorem 2.1 will be given in §5. We emphasize that the notation used above and in the sections which follow is that used in [3]. We also emphasize that in what follows, F, $F|\sigma_a = f_1$, ϕ_1, $B(f_1(Q), r)$, JS_Q, and F^e are all fixed.

3. **Canonical reductions of good C^0 approximations to $F|\sigma_a$.** Given a homeomorphism $g: \sigma_a \to E$, the problem of extending g to $\sigma_a \cup JS$ is called the problem $[g, \sigma_a]$. As in §2, we may associate with the problem $[g, \sigma_a]$ a "reduced" problem $[\phi, \sigma_a]$, where ϕ reduces to the identity on a neighborhood of the "x^n-pole" Q of S, and whose solution implies a solution of $[g, \sigma_a]$. We will now prove a lemma which will aid us in setting up a canonical "explicit solution" for problems $[g, \sigma_a]$ in which g is a good C^0 approximation to $F|\sigma_a$. We state the lemma in a form which will be useful in connection with a later problem of §9. Recall from §2 that $f_1 = F|\sigma_a$ and $B(f_1(Q), r)$ is such that $\bar{B}(f_1(Q), 2r) \subset f_1(\sigma_a)$. Choose a number r' such that $2r < r'$ and $\bar{B}(f_1(Q), r') \subset f_1(\sigma_a)$. Set $\mathscr{M} = f_1^{-1}(\beta B(f_1(Q), r'))$. Note that $J\mathscr{M} \subset \sigma_a$.

LEMMA 3.1. *Given the problem $[f_1, \sigma_a]$ with the associated reduced problem $[\phi_1, \sigma_a]$, where ϕ_1 reduces to the identity on the neighborhood $N_Q^1 = f_1^{-1}(B(f_1(Q), r))$ of JS_Q, let $\mathscr{M}$ be chosen as above. Then there exists a positive number ρ such that if $f_2: \sigma_a \to E$ is any homeomorphism which is a C^0 ρ-approximation to f_1 on $J\mathscr{M}$, then there exists a reduced problem $[\phi_2, \sigma_a]$ of $[f_2, \sigma_a]$, where ϕ_2 reduces to the identity on a neighborhood N_Q^2 of JS_Q.*

Proof. We will merely outline a proof of the lemma. Setting $\rho = (r'/2) - r$, one verifies with the aid of Theorem 3.3 of Part I that if $f_2: \sigma_a \to E$ is any homeomorphism which is a C^0 ρ-approximation to f_1 on $\mathscr{M}$, then

$$\bar{B}(f_1(Q), 2r) \cup \bar{B}(f_2(Q), 2r) \subset f_1(J\mathscr{M}) \cap f_2(J\mathscr{M}). \tag{3.1}$$

By further conditioning ρ, we may require (cf. (3.1) and Proposition 3.1 of Part I) that if $f_2: J\mathscr{M} \to E$ is any homeomorphism which is a C^0 ρ-approximation to f_1 in $\mathscr{M}$, then

$$JS_Q \subset f_2^{-1}(B(f_2(Q), r)). \tag{3.2}$$

We now define a canonical C^∞-diffeomorphism μ_2 of $B(f_2(Q), 2r)$ onto E. Let t_{21} denote the translation of E onto itself defined by

$$t_{21}(y) = y + f_1(Q) - f_2(Q) \qquad (y \in E). \tag{3.3}$$

Now define $\mu_2: B(f_2(Q), 2r) \to E$ by

$$\mu_2(y) = t_{21}^{-1}\mu_1 t_{21}(y) \qquad (y \in E), \tag{3.4}$$

where μ_1 is as defined in §2. We see from (3.3) that $t_{21}(B(f_2(Q), r)) = B(f_1(Q), r)$, and hence by (2.1), we have

$$(3.5) \qquad \mu_2(y) = y \qquad (y \in B(f_2(Q), r)).$$

Setting

$$(3.6) \qquad \eta_2(y) = f_2^{-1}\mu_2^{-1} \qquad (y \in E),$$

we define $\phi_2 : \sigma_a \to E$ by

$$(3.7) \qquad \phi_2(x) = \eta_2 f_2(x) \qquad (x \in \sigma_a).$$

then by (3.5), (3.6), and (3.7), we have

$$(3.8) \qquad \phi_2(x) = x \qquad (x \in f_2^{-1}(B(f_2(Q), r))).$$

Now, by (3.2), $f_2^{-1}(B(f_2(Q), r))$ is a neighborhood of JS_Q. Therefore, setting $N_Q^2 = f_2^{-1}(B(f_2(Q), r))$, we see that Lemma 3.1 is satisfied.

In what follows, if $f_2 : \sigma_a \to E$ is a C^0 ρ-approximation to f_1 on JM, then the reduced problem $[\phi_2, \sigma_a]$ constructed as above will be called *canonically associated* with $[f_2, \sigma_a]$.

LEMMA 3.2. *Corresponding to an arbitrary positive number ε, there exists a positive number $\delta \leqq \rho$ such that if $f_2 : \sigma_a \to E$ is any homeomorphism which is a C^0 δ-approximation to f_1 on σ_a, then the reduced problem $[\phi_2, \sigma_a]$ canonically associated with $[f_2, \sigma_a]$ is such that ϕ_2 is an ε-approximation to ϕ_1 on σ_a.*

Proof. The proof involves a detailed analysis of the canonically associated problem $[\phi_2, \sigma_a]$ constructed above, and will be omitted.

4. Approximation theorems for mappings reducing to the identity on JS_Q.

As previously noted, our notation is that of [3]. In [3], a single mapping $\phi : \sigma_a \to E$ is considered, and an extension Λ_ϕ constructed. We will deal now with the fixed mapping $\phi_1 : \sigma_a \to E$ of §2, and consider a variable mapping $\phi_2 : \sigma_a \to E$ which, like ϕ_1, reduces to the identity on a neighborhood of JS_Q. Corresponding to the mappings ϕ_1 and ϕ_2, the mappings and sets which arise in an analogous manner to those constructed in [3], relative to ϕ, will be given the subscripts 1 and 2 respectively.

Note that if $\phi_2 : \sigma_a \to E$ reduces to the identity on a neighborhood N_Q^2 of JS_Q, then the same mappings t, R, T, a, and sets K, D, M, H', etc., that were used for the construction of Λ_{ϕ_1} can also be used in the construction of a homeomorphism $\Lambda_{\phi_2} : \sigma_a \cup JS \to E$ which extends $\phi_2 \mid [N \cup (\sigma_a - JS)]$. We call the mapping Λ_{ϕ_2}, constructed in this way, the canonical extension of $\phi_2 \mid [N \cup (\sigma_a - JS)]$.

In §2 of [3], a mapping ω of $K - \Theta$ was constructed, and a solution to the problem of finding Λ_ϕ was shown to be implied by the finding of an explicit mapping λ_ω of H' into E which extends $\omega \mid (H' - \Omega)$, where Ω is a suitable chosen compact subset of H'. The mappings R, T, a and the sets, K, D, M, H', etc., used in the construction of λ_{ω_1} can also be used in the construction of λ_{ω_2} where $\omega_2: K-\Theta \to E$ arises from a ϕ_2 of the type described above. The mapping λ_{ω_2} constructed in this way will be called the canonical extension of $\omega_2 \mid (H'-\Omega)$. We will now prove a lemma which will be fundamental in our verification of Theorem 2.1.

LEMMA 4.1. *Given the homeomorphism ω_1 of $K-\Theta$ into E, and the mapping λ_{ω_1} of H' into E which is an extension of $\omega_1 \mid (H'-\Omega)$, let ε be an arbitrary positive number. Then there exists a positive number δ such that if ω_2 is any homeomorphism of $K-\Theta$ into E where ω_2 arises from a mapping $\phi_2: \sigma_a \to E$ as above, and ω_2 is a C^0 δ-approximation to ω_1, then the canonical extension λ_{ω_2} of $\omega_2 \mid (H'-\Omega)$ is a C^0 ε-approximation to λ_{ω_1} on H'.*

Lemma 4.1 will be shown to be a consequence of the two lemmas which follow. Two conventions will be followed in affixing the subscripts 1 and 2 as previously indicated. We will denote the mapping analogous to ω_e, by ω_1^e and ω_2^e. And the mappings T_r and T_r^{-1} $(r=1, 2, \ldots)$ which are used in constructing the explicit extension Λ_{ϕ_1} of ϕ_1, as well as the canonical extension Λ_{ϕ_2} of ϕ_2, will, to avoid possible confusion, now be denoted instead by T^r and T^{-r}, respectively. Since we are dealing with the *canonical* construction of Λ_{ϕ_2}, the mappings ω_2^e, σ_2, etc., are explicit and well defined.

LEMMA 4.2. *Corresponding to an arbitrary positive number ε, there exists a positive number δ, such that if ω_2 is a C^0 δ-approximation to ω_1 on $K-\Theta$, then σ_2 is a C^0 ε-approximation to σ_1 on CG'.*

Proof. We begin by recalling some sets and mappings defined in §§3–5 of [3]. E admits the partition

$$E = \bigcup_{r=0}^{\infty} R^r(K) \cup A \cup P \qquad (P = (8, 0, \ldots, 0)) \tag{4.1}$$

provided A is suitably chosen. A set M is defined by the disjoint union,

$$M = \bigcup_{r=0}^{\infty} R^r(K-\Theta) \cup A. \tag{4.2}$$

And the mappings ω_i^e $(i=1, 2)$ are defined by

$$\omega_i^e(x) = R^r \omega_i R^{-r}(x) \qquad (x \in R^r(K-\Theta),\ i=1, 2) \tag{4.3}$$

and

$$\omega_i^e(x) = x \qquad (x \in A,\ i = 1, 2). \tag{4.4}$$

From the definition (cf. §3 of [3]) of the mappings R^r $(r=0, 1, \ldots)$, it is clear that

$$d(R^r(x_1), R^r(x_2)) \leqq d(x_1, x_2) \qquad (x_1, x_2 \in E,\ r = 0, 1, \ldots). \tag{4.5}$$

Hence from (4.2), (4.3), (4.4), and (4.5), we see that if ω_2 is a C^0 ε-approximation to ω_1 on $K-\Theta$, then ω_2^e is a C^0 ε-approximation to ω_1^e on the set M. Also, as a consequence of the definition of σ_i, $i=1, 2$, (cf. §7 of [3]), we have

$$\sigma_i|T^r(K) = R^{r-1}(\sigma_i|T(K))(R^{r-1})^{-1} \qquad (i = 1, 2, r = 1, 2, \ldots). \tag{4.6}$$

From (4.5) and (4.6), we see that if σ_2 is a C^0 ε-approximation to σ_1 on $T(K)$, then σ_2 is a C^0 ε-approximation to σ_1 on the sets $T^r(K)$, $r=1, 2, \ldots$. Moreover,

$$\sigma_i(x) = \omega_i^e(x) \qquad \left(i = 1, 2, x \in CG' - \bigcup_{r=1}^{\infty} T^r(K)\right), \tag{4.7}$$

hence to prove Lemma 4.2, it suffices to prove the following lemma.

LEMMA 4.3. *Corresponding to an arbitrary positive number ε, there exists a positive number δ, such that if ω_2 is a C^0 δ-approximation to ω_1 on $K-\Theta$, then σ_2 is a C^0 ε-approximation to σ_1 on $T(K)$.*

Proof. Using the uniform continuity of the mappings ω_1^e, $T|K$, and $\omega_1^{-1}|\omega_1(K-G)$, we may successively choose positive constants δ_1, δ_2, and δ_3 such that the following implications hold:

$$d(x_1, x_2) < \delta_1 \Rightarrow d(\omega_1^e(x_1), \omega_1^e(x_2)) < \varepsilon/2 \qquad (x_1, x_2 \in E), \tag{4.8}$$

$$d(x_1, x_2) < \delta_2 \Rightarrow d(T(x_1), T(x_2)) < \delta_1 \qquad (x_1, x_2 \in K), \tag{4.9}$$

$$d(z_1, z_2) < \delta_3 \Rightarrow d(\omega_1^{-1}(z_1), \omega_1^{-1}(z_2)) < \delta_2 \qquad (z_1, z_2 \in \omega_1(K-G)). \tag{4.10}$$

Set

$$\delta = \tfrac{1}{2} \min (E, \delta_3, d(\omega_1(\beta L'), \omega_1(\beta G'))), d(\omega_1(\beta L''), \omega_1(\beta G'')). \tag{4.11}$$

We claim that δ satisfies the requirements of Lemma 4.3.

We show first that if ω_2 is a C^0 δ-approximation to ω_1 on $K-\Theta$, then statements (α), (β), and (γ) below are satisfied. (α) *α_2^1 is a C^0 δ_1-approximation to α_1^1 on $T(K-\mathcal{G}_1) \cap T(K-\mathcal{G}_2)$.*

To verify (α), recall from (6.12) of [3] that

$$\alpha_i^1(x) = T\omega_i^{-1}T^{-1}(x) \qquad (i = 1, 2, x \in T(K-\mathcal{G})). \tag{4.12}$$

Now by Proposition 3.1 of Part I, (4.10), and our choice of δ_1 we have

$$d(\omega_1^{-1}T^{-1}(z), \omega_2^{-1}T^{-1}(z)) < \delta_2 \qquad (z \in T(K-\mathcal{G}_1) \cap T(K-\mathcal{G}_2)). \tag{4.13}$$

Hence by (4.9) and (4.12), we have for $z \in T(K-\mathcal{G}_1) \cap T(K-\mathcal{G}_2)$,

$$d(\alpha_1^1(z), \alpha_2^1(z)) = d(T(\omega_1^{-1}T^{-1}(z)), T(\omega_2^{-1}T^{-1}(z))) < \delta_1. \tag{4.14}$$

Hence (α) is verified.

(β) *$\omega_2^e\alpha_2^1$ is a C^0 ε-approximation to $\omega_1^e\alpha_1^1$ on $T(K-\mathcal{G}_1) \cap T(K-\mathcal{G}_2)$.*

For let $z \in T(K-\mathcal{G}_1) \cap T(K-\mathcal{G}_2)$. Since $\delta < \varepsilon/2$, ω_2^e is a C^0 $\varepsilon/2$-approximation to

ω_1^ε on the set M, which contains the set $T(K-G)=\alpha_i^1(T(K-\mathscr{G}_i))$, $i=1, 2$. From (4.8) and (α) we have

$$d(\omega_1^\varepsilon\alpha_1^1(z), \omega_2^\varepsilon\alpha_2^1(z)) \leqq d(\omega_1^\varepsilon\alpha_1^1(z), \omega_1^\varepsilon\alpha_2^1(z))+d(\omega_1^\varepsilon\alpha_2^1(z), \omega_2^\varepsilon\alpha_2^1(z))$$
$$< \varepsilon/2+\varepsilon/2 = \varepsilon,$$

and (β) is verified.

(γ) $$T(\mathscr{L}_1) \cap T(\mathscr{L}_2) \supset T(\mathscr{G}_1) \cup T(\mathscr{G}_2).$$

Now since $\delta < d(\omega_1(\beta L'), \omega_1(\beta G'))$, we have by Theorem 3.3 of Part I,

$$\mathscr{L}_2' = J\omega_2(\beta L') \supset J\omega_1(\beta G') = \mathscr{G}_1' \tag{4.15}$$

and

$$\mathscr{L}_1' = J\omega_1(\beta L') \supset J\omega_2(\beta G') = \mathscr{G}_2'. \tag{4.16}$$

Since $\mathscr{L}_1' \supset \mathscr{G}_1'$ and $\mathscr{L}_2' \supset \mathscr{G}_2'$, we see that

$$\mathscr{L}_1' \cap \mathscr{L}_2' \supset \mathscr{G}_1' \cup \mathscr{G}_2'. \tag{4.17}$$

A similar argument, using the fact that $\delta < d(\omega_1(\beta L''), \omega_1(\beta G''))$, shows that

$$\mathscr{L}_1'' \cap \mathscr{L}_2'' \supset \mathscr{G}_1'' \cup \mathscr{G}_2''. \tag{4.18}$$

Now $\mathscr{L}_1 = \mathscr{L}_1' \cup \mathscr{L}_1''$, $\mathscr{L}_2 = \mathscr{L}_2' \cup \mathscr{L}_2''$. Hence from (4.17) and (4.18), we have

$$\mathscr{L}_1 \cap \mathscr{L}_2 = (\mathscr{L}_1' \cup \mathscr{L}_1'') \cap (\mathscr{L}_2' \cup \mathscr{L}_2'') \supset \mathscr{G}_1' \cup \mathscr{G}_2' \cup \mathscr{G}_1'' \cup \mathscr{G}_2'' = \mathscr{G}_1 \cup \mathscr{G}_2. \tag{4.19}$$

Note that (γ) follows from (4.19).

We now show that (β) and (γ) imply Lemma 4.3. For $i=1, 2$, and $x \in T(K)$,

$$\begin{aligned} \sigma_i(x) &= \omega_i^\varepsilon\alpha_i^1(x) && (x \in T(K-\mathscr{G}_i)) \\ &= RT^{-1}(x) && (x \in T(\mathscr{L}_i')) \\ &= T^{-1}(x) && (x \in T(\mathscr{L}_i'')). \end{aligned} \tag{4.20}$$

Now if $x \in T(K-\mathscr{G}_1) \cap T(K-\mathscr{G}_2)$, then by (4.20), $\sigma_i(x)=\omega_i^\varepsilon\alpha_i^1(x)$, $i=1, 2$, hence (β) implies that $d(\sigma_i(x), \sigma_2(x)) < \varepsilon$. Now suppose $x \in T(K)-(T(K-\mathscr{G}_1) \cap T(K-\mathscr{G}_2)) = T(\mathscr{G}_1) \cup T(\mathscr{G}_2)$. Then by ($\gamma$) and (4.20), we see that $\sigma_1(x)=\sigma_2(x)$. This completes the proof of Lemma 4.3.

Proof of Lemma 4.1. Let the positive number ε be given.

(i) By the uniform continuity of ω_1 on $K-\Theta$, there exists a positive number δ_1 such that if $x_1, x_2 \in K-\Theta$ and $d(x_1, x_2) < \delta_1$, then $d(\omega_1(x_1), \omega_1(x_2)) < \varepsilon/2$.

(ii) By Proposition 2.1 of Part I and the uniform continuity of σ_1^{-1} on $\sigma_1(D-G')$, there exists a positive number δ_2 such that if σ_2 is a C^0 δ_2-approximation to σ_1 on $D-G'$, then σ_2^{-1} is a C^0 δ_1-approximation to σ_1^{-1} on $\sigma_1(D-G') \cap \sigma_2(D-G')$.

(iii) By Lemma 4.2, corresponding to the positive number δ_2, there exists a positive number δ_3 such that if ω_2 is a C^0 δ_3-approximation to ω_1 on $K-\Theta$, then σ_2 is a C^0 δ_2-approximation to σ_1 on CG'.

Set

$$\delta = \tfrac{1}{2}\min(\varepsilon, \delta_3, d(\omega_1, (\beta L'), \omega_1(\beta G'))). \tag{4.21}$$

One verifies, in a manner similar to the proof of Lemma 4.3, that δ satisfies the requirements of Lemma 4.1.

COROLLARY 4.1. *Given ϕ_1 and Λ_{ϕ_1}, let ε be an arbitrary positive number. Then there exists a positive number δ such that if $\phi_2: \sigma_a \to E$ is any homeomorphism reducing to the identity on a neighborhood N_Q^2 of JS_Q, and such that ϕ_2 is a C^0 δ-approximation to ϕ_1 on σ_a, then the canonical extension Λ_{ϕ_2} of $\phi_2 \mid [N \cup (\sigma_a - JS)]$ is a C^0 ε-approximation to Λ_{ϕ_2} on $\sigma_a \cup JS$.*

The proof is omitted.

5. **Proof of Theorem 2.1.** Corresponding to ε, we want to find a positive number δ satisfying the requirements of Theorem 2.1. We set $\delta_1 = \rho$, where the number ρ is that appearing in Lemma 3.1. Then if $f_2: \sigma_a \to E$ is a C^0 δ_1-approximation to f_1 on $J\mathcal{M}$, we may construct the reduced problem $[\phi_2, \sigma_2]$ which is canonically associated with $[f_2, \sigma_a]$. In what follows, it will be assumed that f_2 is a C^0 δ_1-approximation to f_1 on σ_a.

Choose a positive number s, where $r < s < 2r$, and

$$J\eta_1^{-1} f_1(S) = Jf_1\Lambda_{\phi_1}(S) \subset B(f_1(Q), s) \subset \bar{B}(f_1(Q), s) \subset B(f_1(Q), 2r). \tag{5.1}$$

Set

$$\lambda = d(f_1\Lambda_{\phi_1}(S), \beta B(f_1(Q), s)). \tag{5.2}$$

Using the uniform continuity of $f_1 | J\mathcal{M}$, corresponding to λ, there exists a positive number δ_2 such that

$$d(x_1, x_2) < \delta_2 \Rightarrow d(f_1(x_1), f_1(x_2)) < \lambda/3 \qquad (x_1, x_2 \in J\mathcal{M}). \tag{5.3}$$

By Lemma 3.2 and Corollary 4.1, there exists a positive number $\delta_3 < \lambda/3$ such that if $f_2: \sigma_a \to E$ is a C^0 δ_3-approximation to f_1 on σ_a, then Λ_{ϕ_2} is a C^0 δ_2-approximation to Λ_{ϕ_1} on $\sigma_a \cup JS$. For such a mapping f_2, it can be verified that

$$f_1\Lambda_{\phi_2}(JS) \cup f_2\Lambda_{\phi_2}(JS) \cup t_{21}f_2\Lambda_{\phi_2}(JS) \subset \bar{B}(f_1(Q), s), \tag{5.4}$$

where we note by (3.18) that

$$\Lambda_{\phi_1}(JS) \cup \Lambda_{\phi_2}(JS) \subset J\mathcal{M}. \tag{5.5}$$

Using the uniform continuity of η_1 on the compact set $\bar{B}(f_1(Q), s)$, corresponding to ε, as given in Theorem 2.1, there exists a positive number δ_4 such that

$$d(y_1, y_2) < \delta_4 \Rightarrow d(\eta_1(y_1), \eta_1(y_2)) < \varepsilon/4 (y_1, y_2 \in \bar{B}(f_1(Q), s)). \tag{5.6}$$

Using the uniform continuity of f_1 on $J\mathscr{M}$, there exists a positive number ξ such that

(5.7) $$d(x_1, x_2) < \xi \Rightarrow d(f_1(x_1), f_1(x_2)) < \delta_4 \qquad (x_1, x_2 \in J\mathscr{M}).$$

Using Corollary 4.1, there exists a positive number ζ such that if $[\phi_2, \sigma_a]$ is the reduced problem canonically associated with $[f_2, \sigma_a]$, and ϕ_2 is a C^0 ζ-approximation to ϕ_1 on σ_a, then Λ_{ϕ_2} is a C^0 ξ-approximation to Λ_{ϕ_1} on $\sigma_a \cup JS$.

Finally, using Lemma 3.2, corresponding to ζ, there exists a positive number δ_5 such that if $f_2 \colon \sigma_a \to E$ is C^0 δ_5-approximation to f_1 on σ_a, then the reduced problem $[\phi_2, \sigma_a]$ canonically associated with $[f_2, \sigma_a]$ is such that ϕ_2 is a C^0 ζ-approximation to ϕ_1 on σ_a. Set

(5.8) $$\delta = \min (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \varepsilon/4).$$

We claim that δ satisfies the requirements of Theorem 2.1. Now $F^e = \eta_1 f_1 \Lambda_{\phi_1}$, $F_2^e = \eta_2 f_2 \Lambda_{\phi_2}$ are homeomorphisms of $\sigma_a \cup JS$ which extend $f_1 \mid [N \cup (\sigma_a - JS)]$, $f_2 \mid [N \cup (\sigma_a - JS)]$ respectively. Since $\delta < \varepsilon$, F_2^e is C^0 ε-approximation to F^e on $N \cup (\sigma_a - JS)$. Hence we need only check that F_2^e is a C^0 ε-approximation to F^e on JS. Now for $x \in JS$,

(5.9) $$d(F^e(x), F_2^e(x)) = d(\eta_1 f_1 \Lambda_{\phi_1}(x), \eta_2 f_2 \Lambda_{\phi_2}(x)).$$

Now $\delta \leqq \delta_1, \delta_2, \delta_3$. Hence by (5.5), $\Lambda_{\phi_1}(x)$ and $\Lambda_{\phi_2}(x)$ are both points of $J\mathscr{M}$. And further, (5.1) and (5.4) imply that $f_1 \Lambda_{\phi_1}(x)$ and $f_2 \Lambda_{\phi_2}(x)$ are both points of $\bar{B}(f_1(Q), s)$. By (5.4), $\eta_1 f_1 \Lambda_{\phi_2}(x)$ is defined, and we have from (5.9),

(5.10) $$\begin{aligned} d(F^e(x), F_2^e(x)) \leqq\ & d(\eta_1 f_1 \Lambda_{\phi_1}(x), \eta_1 f_1 \Lambda_{\phi_2}(x)) \\ & + d(\eta_1 f_1 \Lambda_{\phi_2}(x), \eta_2 f_2 \Lambda_{\phi_2}(x)). \end{aligned}$$

Since $\delta \leqq \delta_5$, $d(\Lambda_{\phi_1}(x), \Lambda_{\phi_2}(x)) < \xi$. Hence by (5.5) and (5.7), we see that $d(f_1 \Lambda_{\phi_1}(x), f_1 \Lambda_{\phi_2}(x)) < \delta_4$. Now using (5.1), (5.4), and (5.6), we have

(5.11) $$d(\eta_1 f_1 \Lambda_{\phi_1}(x), \eta_1 f_1 \Lambda_{\phi_2}(x)) < \varepsilon/4 \qquad (x \in JS).$$

Since $\delta \leqq \min (\delta_4, \varepsilon/4)$, we have

(5.12) $$d(x, t_{21}(x)) < \min (\delta_4, \varepsilon/4), d(x, t_{21}^{-1}(x)) < \min (\delta_4, \varepsilon/4) \qquad (x \in E).$$

Also, we have

(5.13) $$\begin{aligned} d(\eta_1 f_1 \Lambda_{\phi_2}(x), \eta_2 f_2 \Lambda_{\phi_2}(x)) &= d(\eta_1 f_1 \Lambda_{\phi_2}(x), t_{21}^{-1} \eta_1 t_{21} f_2 \Lambda_{\phi_2}(x)) \\ &\leqq d(\eta_1 f_1 \Lambda_{\phi_2}(x), \eta_1 t_{21} f_2 \Lambda_{\phi_2}(x)) \\ &\quad + d(\eta_1 t_{21} f_2 \Lambda_{\phi_2}(x), t_{21}^{-1} \eta_1 t_{21} f_2 \Lambda_{\phi_2}(x)), \end{aligned}$$

where, by (5.4), $t_{21} f_2 \Lambda_{\phi_2}(x)$ is a point of $\bar{B}(f_1(Q), s)$. Since $\delta < \varepsilon/4$, by (5.12) we see that the last term of (5.13) is less than $\varepsilon/4$. For the first term of (5.13), we have

(5.14) $$\begin{aligned} d(\eta_1 f_1 \Lambda_{\phi_2}(x), \eta_1 t_{21} f_2 \Lambda_{\phi_2}(x)) \leqq\ & d(\eta_1 f_1 \Lambda_{\phi_2}(x), \eta_1 f_2 \Lambda_{\phi_2}(x)) \\ & + d(\eta_1 f_2 \Lambda_{\phi_2}(x), \eta_1 t_{21} f_2 \Lambda_{\phi_2}(x)). \end{aligned}$$

Since $\delta < \min(\delta_2, \delta_3, \delta_4, \delta_5)$ we see that $f_1\Lambda_{\phi_2}(x), f_2\Lambda_{\phi_2}(x)$ are points of $\bar{B}(f_1(Q), s)$ such that $d(f_1\Lambda_{\phi_2}(x), f_2\Lambda_{\phi_2}(x)) < \delta_4$, and hence, using (5.6),

$$d(\eta_1 f_1\Lambda_{\phi_2}(x), \eta_1 f_2\Lambda_{\phi_2}(x)) < \varepsilon/4.$$

Also, by (5.4) and (5.12), $f_2\Lambda_{\phi_2}(x)$ and $t_{21}f_2\Lambda_{\phi_2}(x)$ are points of $\bar{B}(f_1(Q), s)$ such that $d(f_2\Lambda_{\phi_2}(x), t_{21}f_2\Lambda_{\phi_2}(x)) < \delta/4$. Hence, using (5.6), $d(\eta_1 f_2\Lambda_{\phi_2}(x), \eta_1 t_{21}f_2\Lambda_{\phi_2}(x)) < \varepsilon/4$. Combining these results, we have

$$d(\eta_1 f_1\Lambda_{\phi_2}(x), \eta_2 f_2\Lambda_{\phi_2}(x)) < \varepsilon/4+\varepsilon/4+\varepsilon/4 = 3\varepsilon/4 \qquad (x \in JS). \tag{5.15}$$

Then by (5.10), (5.11), and (5.15), we have

$$d(F^e(x), F_2^e(x)) < \varepsilon/4+3\varepsilon/4 = \varepsilon \qquad (x \in JS). \tag{5.16}$$

This completes the proof of Theorem 2.1.

6. **Proof of Theorem 1.1.** Let ε be an arbitrary positive number. We want to find a positive number δ satisfying Theorem 1.1. Let F^e denote the homeomorphism of $\sigma_a \cup JS$ into E which extends $F \mid [N_1 \cup (\sigma_a - JS)]$, where N_1 is a neighborhood of S, and where F^e is constructed as in §2. Note that

$$F(\sigma_a \cup JS) = F^e(\sigma_a \cup JS). \tag{6.1}$$

Hence the mapping $F(F^e)^{-1}\colon F^e(\sigma_a \cup JS) \to E$ is a homeomorphism of $F^e(\sigma_a \cup JS)$ onto itself, and reduces to the identity on the set $F^e(N_1 \cup (\sigma_a - JS))$. Therefore, we may extend $F(F^e)^{-1}$ to a homeomorphism H of E onto itself by setting

$$H(y) = y \qquad (y \in CJF^e(S) \cup F^e(N_1)) \tag{6.2}$$

$$H(y) = F(F^e)^{-1} \qquad (y \in F^e(\sigma_a \cup JS)). \tag{6.3}$$

The mapping H is uniformly continuous, and therefore corresponding to ε, there exists a positive number λ such that

$$d(y_1, y_2) < \lambda \Rightarrow d(H(y_1), H(y_2)) < \varepsilon \qquad (y_1, y_2 \in E). \tag{6.4}$$

Let b be a constant such that $0 < b < a$ and $\sigma_b \subset N_1$. Set

$$\delta_1 = \min d(F^e(S(c,1+b)), F^e(S(c,1+b/2))), d(F^e(S(c,1-b)), F^e(S(c, 1-b/2))). \tag{6.5}$$

Now if $g\colon \sigma_a \to E$ is any homeomorphism which is a C^0 δ-approximation to $F|\sigma_a$, then by Theorem 3.3 of Part I,

$$Jg(S(c, 1+b/2)) \subset JF^e(S(c, 1+b)) \tag{6.6}$$

and

$$Jg(S(c, 1-b/2)) \supset JF^e(S(c, 1-b)). \tag{6.7}$$

As a consequence of (6.6) and (6.7), we have

$$g(\sigma_{b/2}) \subset F^e(\sigma_b) \subset F^e(N_1). \tag{6.8}$$

Relation (6.7) implies that

$$g(\sigma_a - JS) \subset CJF^e(S) \cup F^e(N_1). \tag{6.9}$$

By Theorem 2.1, corresponding to λ, there exists a positive number δ_2 such that if $g\colon \sigma_a \to E$ is any homeomorphism which is a C^0 δ_2-approximation to $F|\sigma_a$, then there exists a homeomorphism $G^e\colon \sigma_a \cup JS \to E$, which extends $g \mid [N_1 \cup (\sigma_a - JS)]$ and is such that G^e is a C^0 λ-approximation to F^e. Set

$$\delta = \min(\delta_1, \delta_2), \tag{6.10}$$

$$N = \sigma_{b/2}, \tag{6.11}$$

and

$$G = HG^e. \tag{6.12}$$

One verifies that δ, N and G satisfy the requirements of Theorem 1.1.

Remark. It is an open question whether Theorem 1.1 remains true if ε is taken to be an arbitrary modulus on $(\sigma_a \cup JS) - c$. If Theorem 1.1 could be so strengthened, then (cf. Lemma 4.3 of [8] and Theorem 5 of [11]) the following would hold: there would exist a positive number δ such that if $g\colon \sigma_a \to E$ is any homeomorphism which is a C^0 δ-approximation to $F|\sigma_a$, then $F|S$ is extendable to a *stable* homeomorphism of E, so is $g|S$, and vice versa.

We state the following theorem without proof. It can be established with the aid of Theorem 1.1 and its proof.

Theorem 6.1. *Let $f_i\colon \sigma_a \to E$, $i = 1, 2, \ldots$ be homeomorphisms such that* $\lim_{i\to\infty} f_i(x) = F(x)$ $(x \in \sigma_a)$, *where $F\colon \sigma_a \cup JS \to E$ is some given homeomorphism. Then there exists, for each i, a homeomorphism $F_i\colon \sigma_a \cup JS \to E$ extending $f_i \mid [N \cup (\sigma_a - JS)]$ where N is a neighborhood of S, and such that* $\lim_{i\to\infty} F_i(x) = F(x)$ $(x \in \sigma_a \cup JS)$.

7. **Theorem 1.1 in the differentiable case.** Suppose $F\colon \sigma_a \cup JS \to E$ is a C^k-diffeomorphism, $(k \geqq 0)$. Then as a consequence of Theorem 2.1 of [3], the homeomorphism F^e considered in §§2–6 is a C^k_z-diffeomorphism, where z is a point of JS. Similarly, if $g\colon \sigma_a \to E$ is a C^p-diffeomorphism, the canonical extension G^e constructed to satisfy Theorem 2.1 is a C^p_z-diffeomorphism. Now as a consequence of Lemma 3.5 of [8], we may assume that $G^e(z) = F^e(z)$, where G^e still satisfies Theorem 2.1. Hence we see that G as in (6.12) is a C^m_z-diffeomorphism, where $M = \min(k, p)$.

An interesting unsolved question is the following: if $g\colon \sigma_a \to E$ is a C^p-diffeomorphism, and a sufficiently good C^0 approximation to $F|\sigma_a$, does there exist a homeomorphism $G\colon \sigma_a \cup JS \to E$ satisfying Theorem 1.1 and such that G is a C^p_z-diffeomorphism?

In Theorem 1.2 of [8], an arbitrary point z of S, and a C^m-diffeomorphism $g\colon \sigma_a \to E$ are considered. A C^m_z-diffeomorphism $G\colon \sigma_a \cup JS \to E$ (actually G as constructed in [8] maps E onto itself) is constructed which extends $g \mid (N - k_z(S))$, where N is a neighborhood of S, and k_z is a "cone of singular approach" to z with axis interiorly normal to S at z. One may show that if $g: \sigma_a \to E$ is a C^p-diffeomorphism, and a good C^0-approximation to the C^k-diffeomorphism $F|\sigma_a$, then corresponding to an arbitrary point $z \in S$, there exists a homeomorphism $G: \sigma_a \cup JS \to E$ which extends $g \mid (N - k_z(S))$, is a C_z^m-diffeomorphism, where $m = \min(p, k)$, and is a good C^0-approximation to F.

8. **C^1-approximations.** The following theorem, which we state without proof, can be established (cf. Theorem 2.2 of Part I) in a manner analogous to the proof of Theorem 2.1. Again we let $F: \sigma_a \cup JS \to E$ be a C^k-diffeomorphism, $(k \geqq 1)$, and let $F^e: \sigma_a \cup JS \to E$ be the homeomorphism constructed as in §2. Note that F^e is a C_z^k-diffeomorphism, where $z \in \dot{J}S$.

Theorem 8.1. *Corresponding to an arbitrary positive number ε, there exists a neighborhood N of S in σ_a, and a positive number δ, with the following properties. If $g: \sigma_a \to E$ is any C^p-diffeomorphism, $p \geqq 1$, which is a C^1 δ-approximation to $F|\sigma_a$, then there exists a C_z^p-diffeomorphism $G: \sigma_a \cup JS \to E$ which extends*

$$g \mid [N \cup (\sigma_a - JS)],$$

and is a C^1 ε-approximation to F^e on $(\sigma_a \cup JS) - z$.

With the aid of Theorem 8.1 and arguments paralleling those of §7, we could establish a theorem, analogous to Theorem 1.1, for C^1-approximations. However, we have the following even stronger result for C^1-approximations.

Theorem 8.2. *Corresponding to an arbitrary positive number ε, there exists a neighborhood N of S in σ_a, and a positive number δ, with the following properties. If $g: \sigma_a \to E$ is any C^p-diffeomorphism, $p \geqq 1$, which is a C^1 δ-approximation to $F|\sigma_a$, then there exists a C^m-diffeomorphism $G: \sigma_a \cup JS \to E$, where $m = \min(k, p)$, such that G extends $g \mid [N \cup (\sigma_a - JS)]$, G is a C^1 ε-approximation to F, and, in addition,*

$$G \mid (\dot{J}S - \sigma_a) = F \mid (\dot{J}S - \sigma_a).$$

Proof. Let b, e be positive numbers such that $1 - a < b < 1$, $e < 1 - b$. Let $\alpha: R \to R$ be a real valued C^∞-mapping such that $\alpha(t) = 0$ for $t \leqq b$, $0 < \alpha(t) < 1$ for $b < t < 1 - e$, and $\alpha(t) = 1$ for $t \geqq 1 - e$. With the aid of Theorem 3.1 of Part I, one verifies that for δ sufficiently small, the following is true: if $g: \sigma_a \to E$ is any C^p-diffeomorphism, $(p \geqq 1)$, where g is a C^1 δ-approximation to $F|\sigma_a$, then $G: \sigma_a \cup JS \to E$ defined by

$$G(x) = F(x) + \alpha(\|x\|)[g(x) - F(x)] \qquad (x \in \sigma_a)$$
$$G(x) = F(x) \qquad (x \in JS(c, b))$$

satisfies the requirements of Theorem 8.2.

9. **Spreading a small area of approximation.** In the approximation theorems we have previously studied, $F: \sigma_a \cup JS \to E$ was a given homeomorphism, and we considered homeomorphisms $g: \sigma_a \to E$ which were good C^0-approximations to

$F|\sigma_a$ on all of σ_a. The question arises as to what can be said about homeomorphisms $g: \sigma_a \to E$ which approximate $F|\sigma_a$ only on a portion of σ_a, e.g., can we extend such mappings so as to approximate F on a large portion of $\dot{J}S$? Certainly a necessary condition on g for an affirmative answer to this question is that, for example, $\delta(f(S), g(S))$ be sufficiently small. With this in mind, we state the following theorem.

THEOREM 9.1. *Let $F: \sigma_a \cup JS \to E$ be a homeomorphism, let V be some fixed neighborhood in σ_a of a point $Q \in S$, and let A be any compact subset of JS. Corresponding to an arbitrary positive number ε, there exists a positive number η such that for any homeomorphism $g: \sigma_a \to E$ satisfying*

(i) *$g|V$ is a C^0 η-approximation to $F|V$ on V,*

(ii) *$\delta(F(S(0, t)), g(S(0, t))) < \eta$ $(1-a<t<1+a)$,*

the following is true. There exists a homeomorphism $G: \sigma_a \cup JS \to E$, which extends $g \mid [N \cup (\sigma_a - JS)]$. where N is a neighborhood of S, such that G is a C^0 ε-approximation to F on A.

Proof. We may assume without loss of generality that Q is the x^n-pole of S and that $A \subset \dot{J}S(c, 1-a)$. With these assumptions, we first prove Theorem 9.1 in the special case where F=the identity mapping of $\sigma_a \cup JS$. In this special case, the variable homeomorphism of σ_a approximating F will be denoted by h instead of g. Choose a positive constant $b<a$ and positive constants r_0, r_1, r_2, and r_3 such that $r_0<r_1<r_2<r_3$ and

$$(9.1) \qquad JS(Q, 2r_2) \subset \dot{J}S(Q, r_3) \subset JS(Q, r_3) \subset V \cap \sigma_b.$$

Let I denote the identity mapping of $\sigma_b \cup JS$, and let μ_I denote a C^∞-mapping of $B(Q, 2r_2)$ onto E such that μ_I reduces to the identity on $B(Q, r_2)$. Set

$$S_Q = S(Q, r_0), \quad \mathscr{M} = S(Q, r_3), \quad \text{and} \quad \phi_I = \mu_I^{-1}|\sigma_b.$$

Corresponding to ϕ_I, S_Q, and the reflection t in S_Q, we choose the sets K and $K_{\nu d}$, $\nu=1, 2, 3, 4$ as in §2 of [3]. We modify the set H', L', G', Θ' as chosen in §2 of [3] as follows. Choose a positive number e such that $4e<r_0$. Note that the $(n-1)$-spheres $S(c, 1-\nu e)$, $\nu=1, 2, 3, 4$, are subsets of σ_b. Let the open subset of $K_{\nu d}$ on which $x \in t(\dot{J}S(c, 1-\nu e))$ be denoted by H' for $\nu=1$, and by L' for $\nu=2$. Let the closures of the subsets of $K_{\nu d}$ on which $x \in t(JS(c, 1-\nu e))$ be denoted by G' for $\nu=3$, and by Θ' for $\nu=4$. We now set $\omega_I = \Psi_I \mid (K-\Theta)$ and construct an explicit solution of $\omega_I \mid (H'-\Omega)$ as before. The contraction a of D onto H' leaving L' pointwise invariant certainly exists with H' and L' rechosen as above. Also, we may assume that $H' \subset Z'$, where Z' is the open subset of K on which $x^n < -d$. Hence the mapping T chosen as before may also be used with the rechosen H', L', G', Θ'. The form of the explicit extension λ_{ω_I} of ω_I is then completely analogous to that constructed with H', G', L', Θ' chosen originally. Again, using λ_{ω_I} we get a homeomorphism $I^e\colon \sigma_b \cup JS \to E$ which extends $I \mid [N \cup (\sigma_b - JS)]$, where N is a neighborhood of S in E.

Our choice of H', G', L', Θ' is motivated by the data of Theorem 9.1. For suppose $\phi_h\colon \sigma_b \to E$ is any homeomorphism which reduces to the identity on $JS(Q, r_1) \subset JS_Q$, and is such that

$$\delta(\phi_I(S(c, u)), \phi_h(S(c, u))) < \eta \qquad (1-b < u \leqq 1). \tag{9.2}$$

The homeomorphism $\omega_h\colon K - \Theta \to E$ canonically associated with ϕ_h, can be shown to have the fundamental property

$$\delta(\omega_I(\beta G'), \omega_h(\beta G')) < \eta. \tag{9.3}$$

From the nature of $\phi_I = \mu_I^{-1} \mid \sigma_b$, it is clear that $\phi_I(JS(c, u)) \subset JS(c, u)$ for $1-b < u \leqq 1$, and hence

$$\omega_I(L') \subset L'_I \subset L'. \tag{9.4}$$

We note, then, that

$$\lambda_{\omega_I}(y) = y \qquad (y \in \mathscr{G}'_I), \tag{9.5}$$

and therefore

$$\Lambda_{\phi_I}(x) = x \qquad (x \in t(\mathscr{G}'_I) = \phi_I t(G')). \tag{9.6}$$

Using (9.6), we have

$$I^e(x) = \mu_I \Lambda_{\phi_I}(x) = \mu_I(x) \qquad (x \in t(\mathscr{G}'_I)). \tag{9.7}$$

Since $\mu_I^{-1}(JS(c, 1-a)) \subset \mu_I^{-1}(t(G')) = t(\mathscr{G}'_I)$, we see that

$$I^e(t(\mathscr{G}'_I)) = t(G') \supset JS(c, 1-b) \supset JS(c, 1-a). \tag{9.8}$$

Choose a positive constant δ_1 such that if $h\colon \sigma_b \to E$ is any homeomorphism for which $h|J\mathscr{M}$ is a C^0 δ-approximation to $I|J\mathscr{M}$, then there exists (cf. §3) a reduced problem $[\phi_h, \sigma_b]$ canonically associated with $[h, \sigma_b]$, and such that ϕ_h reduces to the identity on $JS(Q, r_1)$. Set

$$\zeta = \min\,(d(\omega_I(\beta G'), \omega_I(\beta L')), d(\omega_I(\beta G'), \omega_I(\beta \Theta'))). \tag{9.9}$$

Then setting $\delta_2 = \min\,(\delta_1, \zeta/4)$, we assert that if $h\colon \sigma_b \to E$ is any homeomorphism which is a C^0 δ_2-approximation to I on $J\mathscr{M}$, and is such that $\delta(h(S(c, u), S(c, u))) < \delta_2$ for $1-b < t \leqq 1$, then the reduced problem $[\phi_h, \sigma_a]$ canonically associated with $[h, \sigma_b]$ is such that

$$\delta(\phi_I(S(c, u)), \phi_h(S(c, u))) < \zeta \qquad (1-b \leqq u \leqq 1). \tag{9.10}$$

To verify (9.10), we denote the translation $x \to x + Q - h(Q)$ by t_h, and note that

$$\begin{aligned} &\delta(\phi_I(S(c, u)), \phi_h(S(c, u))) \\ &\qquad = \delta(\mu_I^{-1}(S(c, u)), h^{-1} t_h^{-1} \mu_I^{-1} t_h h(S(c, u))) \qquad (1-b < u \leqq 1). \end{aligned} \tag{9.11}$$

We observe that $d(t_h(x), x) = d(t_h^{-1}(x), x) < \delta_2$, for $x \in E$, $d(\mu_I^{-1}(x), \mu_I^{-1}(y)) \leqq d(x, y)$ for $x, y \in E$, and $t_h^{-1}\mu_I^{-1}t_h h(S(c, u))$ is a subset of $h(J\mathscr{M})$ on which h^{-1} is a C^0 δ_2-approximation to I. With the aid of these relations, the inequality $\delta(M, M'') \leqq \delta(M, M') + \delta(M', M'')$, and our choice of δ_2, (9.10) follows easily.

Let $\mathscr{N}$ be a topological $(n-1)$-sphere in E such that

$$tJ\beta\mathscr{H}_I' \subset \dot{J}\mathscr{N} \subset J\mathscr{N} \subset B(Q, 2r_2). \tag{9.12}$$

By the uniform continuity of μ_I on the compact set $J\mathscr{N}$, there exists, corresponding to ε, a positive number $\xi < \varepsilon$ such that

$$d(y_1, y_2) < \xi \Rightarrow d(\mu_I(y_1), \mu_I(y_2)) < \varepsilon/2 \qquad (y_1, y_2 \in J\mathscr{N}). \tag{9.13}$$

Set

$$\delta = \min(\delta_2, \xi, d(t(\beta\mathscr{L}_I'), t(\beta\mathscr{H}_I')), d(t(\beta\mathscr{H}_I'), \mathscr{N})). \tag{9.14}$$

Now if $h: \sigma_b \to E$ is any homeomorphism which is a C^0 δ-approximation to I on $J\mathscr{M}$ and is such that $\delta(h(S(c, u)), S(c, u)) < \delta$ for $1 - b < u \leqq 1$, then (9.10) holds. By (9.10) and (9.3) we have

$$\delta(\omega_I(\beta G'), \omega_h(\beta G')) < \zeta. \tag{9.15}$$

Hence, using (9.9) and (9.4), we have

$$\omega_h(G') \subset \omega_I(L') \subset L', \tag{9.16}$$

and

$$\mu_I^{-1}(JS(c, 1-a)) \subset \mu_I^{-1}t(\Theta') = Jt\omega_I(\beta\Theta') \subset Jt\omega_h(\beta G') = t(\mathscr{G}_h'). \tag{9.17}$$

Note that (9.16) implies

$$t(\mathscr{G}_h') \subset t(\mathscr{L}_I') \subset B(Q, 2r_2) \subset J\mathscr{M}. \tag{9.18}$$

It follows from (9.16) that the mapping λ_{ω_h} canonically associated with ω_h is such that $\lambda_{\omega_h}(y) = y$ for $y \in t(\mathscr{G}_h')$, and hence

$$\Lambda_{\phi_h}(x) = x \qquad (x \in t(\mathscr{G}_h')). \tag{9.19}$$

Consequently, by (9.19), the homeomorphism $H^e: \sigma_b \cup JS \to E$ arising from Λ_{ϕ_h} is such that

$$H^e(x) = \mu_h h(x) \qquad (x \in t(\mathscr{G}_h') \subset JM). \tag{9.20}$$

From (9.20) and (9.14), we see that $ht(\mathscr{G}_h') \subset t(\mathscr{H}_I')$, so that, since $d(t_h(x), x) < \delta$ for all $x \in E$, we have

$$t_h ht(\mathscr{G}_h') \subset J\mathscr{M}. \tag{9.21}$$

Thus for $x \in t(\mathscr{G}_h')$, we have $t_h h(x) \in J\mathscr{N}$. And by (9.14) $d(t_h h(x), x) < \xi$. Hence by (9.13),

$$d(\mu_I t_h h(x), \mu_I(x)) < \varepsilon/2 \qquad (x \in t(\mathscr{G}_h')). \tag{9.22}$$

Now from (9.20) and (9.22), we have

$$d(H^e(x), \mu_l(x)) = d(t_h^{-1}\mu_l t_h h(x), \mu_l(x)) \leqq \delta+\varepsilon/2 < \varepsilon/2+\varepsilon/2 = \varepsilon \qquad (x \in t(\mathscr{G}'_h)). \tag{9.23}$$

Then using (9.23), (9.7), (9.8), and (9.17), we have

$$d(H^e(x), I^e(x)) < \varepsilon \qquad (x \in (I^e)^{-1}(JS^{n-1}(c, 1-a))). \tag{9.24}$$

Noting that $(I^e)^{-1}(x)=x$ on a neighborhood of S in E, and setting $H=H^e(I^e)^{-1}$, we obtain a homeomorphism H of $\sigma_b \cup JS$ into E such that H extends $h \mid [N \cup (\sigma_b - JS)]$ for a sufficiently small neighborhood N of S in E. From (9.24), and the fact that $I=I^e(I^e)^{-1}$, we have

$$d(H(x), I(x)) < \varepsilon \qquad (x \in JS^{n-1}(c, 1-a)). \tag{9.25}$$

Now extend H to $\sigma_a \cup JS$ by setting $H(x)=h(x)$ for $x \in \sigma_a - JS$. From (9.25), we see that Theorem 11.1 (with h replacing g) is satisfied by the homeomorphism H (in place of G), in the special case $F=$the identity mapping of $\sigma_a \cup JS$.

We now return to the general case of Theorem 9.1. Suppose, then, that $F: \sigma_a \cup JS \to E$ is an arbitrary homeomorphism. Choosing a positive constant $b<a$ as before, there exists, corresponding to ε as given in Theorem 9.1, a positive number λ such that

$$d(x_1, x_2) < \lambda \Rightarrow d(F(x_1), F(x_2)) < \varepsilon \qquad (x_1, x_2 \in \sigma_b \cup JS). \tag{9.26}$$

From the previous results we see that, corresponding to λ, there exists a positive number ζ such that for any homeomorphism $h: \sigma_b \to E$ which is a C^0 ζ-approximation to the identity on $J\mathscr{M}$ and satisfies $\delta(h(S(c, u)), S(c, u)) < \zeta$ for $1-b<u\leqq 1$, the following is true. There exists a homeomorphism $H: \sigma_b \cup JS \to E$ which extends $h \mid [N \cup (\sigma_b - JS)]$, where N is a neighborhood of S in E, and satisfies $d(H(x), x) < \lambda$ for $x \in JS(c, 1-a)$.

Using the results in §3 of Part I, there exists a positive number $\delta < \varepsilon$ such that any homeomorphism $g: \sigma_a \to E$ which is a C^0 δ-approximation to F on V, and satisfies $\delta(F(S(c, u)), g(S(c, u))) < \delta$ for $1-a<u<1+a$ will have the following four properties: $g(\sigma_b) \subset f(\sigma_a)$ and $JF^{-1}g(S) \subset \sigma_b \cup JS$, while the homeomorphism $h: \sigma_b \to E$ defined by $h=F^{-1}(g|\sigma_b)$ will be a C^0 ζ-approximation to the identity on $J\mathscr{M}$ satisfying $\delta(h(S(c, u)), S(c, u)) < \zeta$ for $1-b<u\leqq 1$. It follows then, from the *special case* of Theorem 9.1 established previously, that there exists a homeomorphism $H: \sigma_b \cup JS \to E$ which extends $h \mid [N \cup (\sigma_b - JS)]$ and is such that

$$d(H(x), x) < \lambda \qquad (x \in JS(c, 1-a)). \tag{9.27}$$

Setting $G=FH$, we obtain a homeomorphism G of $\sigma_b \cup JS$ into E which extends $g \mid [N \cup (\sigma_b - JS)]$. Moreover, using (9.26), (9.27), and the fact that

$$H(JS^{n-1}(c, 1-a)) \subset H(JS) \subset \sigma_b \cup JS,$$

we see that

$$(9.28) \qquad d(F(x), G(x)) = d(F(x), FH(x)) < \varepsilon \qquad (x \in JS(c, 1-a)).$$

Hence G satisfies the requirements of Theorem 9.1.

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Purdue University,
Lafayette, Indiana